ANALYSIS II MIDTERM EXAMINATION

Total marks: 35

(1) State the Substitution Theorem (for integration), and use it to evaluate the integral $\int_1^4 \frac{cos(\sqrt{t})}{\sqrt{t}}dt$. (5 marks)

(2) Let $f : [1,2] \to \mathbb{R}$ be the function defined by $f(x) = x$ for $x \in \mathbb{Q} \cap [1,2]$, and $f(x) = 0$ for $x \in [1,2] - \mathbb{Q}$. Calculate the upper and lower Riemann integrals $U(f)$ and $L(f)$ of f. Is f integrable? (6 marks)

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$. For $n \in \mathbb{N}$, define $f_n(x) = f(x + \frac{1}{n})$ for every $x \in \mathbb{R}$. Does the sequence of functions (f_n) converge uniformly on $\mathbb{R}$? (6 marks)

(4) If the partial sums s_n of the series $\sum_{n=1}^{\infty} a_n$ are bounded, then show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, and $\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$. (6 marks)

(5) Let $f_n(x) =$Arctan(nx). Describe the pointwise limit function f of this sequence. Show that f_n converges to f uniformly on $[a, \infty)$ for any $a > 0$, but the convergence is not uniform on $(0, \infty)$. (6 marks)

(6) If a and b are positive numbers, then prove that the series $\sum_{n=1}^{\infty}(an+b)^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$. (6 marks)

Date: February 19, 2016.